

# Congruences and recursions for the cubic partitions

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**Abstract.** Let  $p_2(n)$  denote the number of cubic partitions. In this paper, we shall present two new congruences modulo 11 for  $p_2(n)$ . We also provide an elementary alternative proof of a congruence established by Chan. Furthermore, we will establish a recursion for  $p_2(n)$ , which is a special case of a broader class of recursions.

**Keywords.** Cubic partition, congruence, recursion.

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## 1. INTRODUCTION

A partition of a natural number  $n$  is a nonincreasing sequence of positive integers whose sum equals  $n$ . Let  $p(n)$  be the number of partitions. Among Ramanujan's discoveries, the following identity

$$\sum_{n \geq 0} p(5n + 4)q^n = 5 \frac{(q^5; q^5)_\infty}{(q; q)_\infty^6}$$

is regarded as his “Most Beautiful Identity” by both Hardy and MacMahon; see [8, p. xxxv]. Here as usual we denote

$$(a; q)_\infty = \prod_{n \geq 0} (1 - aq^n).$$

This identity immediately leads to the following famous congruence

$$p(5n + 4) \equiv 0 \pmod{5}.$$

Ramanujan also discovered two congruences with different moduli, namely,

$$\begin{aligned} p(7n + 5) &\equiv 0 \pmod{7}, \\ p(11n + 6) &\equiv 0 \pmod{11}. \end{aligned}$$

Motivated by Ramanujan's result, Chan [3] introduced the notion of cubic partition of nonnegative integers. Let  $p_2(n)$  be the number of such partitions. Its generating function is given by

$$\sum_{n \geq 0} p_2(n)q^n = \frac{1}{(q; q)_\infty (q^2; q^2)_\infty}, \quad |q| < 1. \quad (1.1)$$

From an identity on the Ramanujan's cubic continued fraction, Chan established the following elegant identity

$$\sum_{n \geq 0} p_2(3n+2)q^n = 3 \frac{(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q; q)_\infty^4 (q^2; q^2)_\infty^4}, \quad (1.2)$$

which immediately implies

$$p_2(3n+2) \equiv 0 \pmod{3}. \quad (1.3)$$

Later on, many authors studied other Ramanujan-like congruences for  $p_2(n)$ . For example, Chan and Lin [4] found four new congruences modulo 7 by using modular forms, whereas Xiong [9] established sets of congruences modulo powers of 5.

We will present two new congruences modulo 11 next in Section 2. Then in Section 3, we will provide an elementary alternative proof of (1.3). At last, we will establish a recursion for  $p_2(n)$ , which is a special case of a broader class of recursions.

## 2. NEW CONGRUENCES MODULO 11 FOR $p_2(n)$

In this section, we shall present two new congruences modulo 11 for  $p_2(n)$ . Unlike previous congruences modulo 5 or 7, the two congruences are of the type  $p_2(297n+t)$ , with 297 not being the square of 11. Our results are

**Theorem 2.1.** *For any nonnegative integer  $n$ ,*

$$p_2(297n+t) \equiv 0 \pmod{11} \quad (2.1)$$

where  $t = 62$  and  $161$ .

To prove the two congruences, we need to use a result of Radu and Sellers [7, Lemma 2.4], which can be tracked back to [6, Lemma 4.5]. Before introducing the result of Radu and Sellers, we will briefly interpret some notations.

Let  $\Gamma := SL_2(\mathbb{Z})$ . For a positive integer  $N$ , the congruence subgroup  $\Gamma_0(N)$  of level  $N$  is defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N} \right\}.$$

It is known that

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}).$$

Moreover, we write

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \in \mathbb{Z} \right\}.$$

For a positive integer  $M$ , let  $R(M) = \{r : r = (r_{\delta_1}, \dots, r_{\delta_k})\}$  be the set of integer sequences indexed by the positive divisors  $1 = \delta_1 < \dots < \delta_k = M$  of  $M$ . Let  $m$  be a positive integer and  $[s]_m$  the set of all elements congruent to  $s$  modulo  $m$ . Let  $\mathbb{Z}_m^*$  denote the set of all invertible elements in  $\mathbb{Z}_m$ , and  $\mathbb{S}_m$  denote the set of all squares in  $\mathbb{Z}_m^*$ . For  $t \in \{0, \dots, m-1\}$ , let  $\overline{\odot}_r$  be the map  $\mathbb{S}_{24m} \times \{0, \dots, m-1\} \rightarrow \{0, \dots, m-1\}$  with

$$([s]_{24m}, t) \mapsto [s]_{24m} \overline{\odot}_r t \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m},$$

and write  $P_{m,r}(t) = \{[s]_{24m} \overline{\odot}_r t \mid [s]_{24m} \in \mathbb{S}_{24m}\}.$

Denote by  $\Delta^*$  the set of tuples  $(m, M, N, t, r = (r_\delta))$  satisfying conditions given in [7, p. 2255]. Let  $\kappa = \kappa(m) = \gcd(m^2 - 1, 24)$ . We set

$$p_{m,r}(\gamma) = \min_{\lambda \in \{0, \dots, m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta(a + \kappa\lambda c), mc)}{\delta m}$$

and

$$p_{r'}^*(\gamma) = \frac{1}{24} \sum_{\delta|N} \frac{r'_\delta \gcd^2(\delta, c)}{\delta}$$

where  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $r \in R(M)$ , and  $r' \in R(N)$ .

Let

$$f_r(q) := \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n \geq 0} c_r(n) q^n$$

for some  $r \in R(M)$ . The lemma of Radu and Sellers is given as follows.

**Lemma 2.2.** *Let  $u$  be a positive integer,  $(m, M, N, t, r = (r_\delta)) \in \Delta^*$ ,  $r' = (r'_\delta) \in R(N)$ ,  $n$  be the number of double cosets in  $\Gamma_0(N) \backslash \Gamma / \Gamma_\infty$  and  $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$  be a complete set of representatives of the double coset  $\Gamma_0(N) \backslash \Gamma / \Gamma_\infty$ . Assume that  $p_{m,r}(\gamma_i) + p_{r'}^*(\gamma_i) \geq 0$  for all  $i = 1, \dots, n$ . Let  $t_{\min} := \min_{t' \in P_{m,r}(t)} t'$  and*

$$v := \frac{1}{24} \left( \left( \sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta \right) [\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta r'_\delta \right) - \frac{1}{24m} \sum_{\delta|M} \delta r_\delta - \frac{t_{\min}}{m}.$$

Then if

$$\sum_{n=0}^{\lfloor v \rfloor} c_r(mn + t') q^n \equiv 0 \pmod{u}$$

for all  $t' \in P_{m,r}(t)$ , then

$$\sum_{n \geq 0} c_r(mn + t') q^n \equiv 0 \pmod{u}$$

for all  $t' \in P_{m,r}(t)$ .

*Proof of Theorem 2.1.* By the binomial theorem and (1.1), one readily sees that

$$\sum_{n \geq 0} p_2(n) q^n \equiv \frac{(q; q)_\infty^{10}}{(q^2; q^2)_\infty (q^{11}; q^{11})_\infty} =: \sum_{n \geq 0} g_{2,11}(n) q^n \pmod{11}. \quad (2.2)$$

We first consider the case of  $p_2(297n + 62)$ . Setting

$$(m, M, N, t, r = (r_1, r_2, r_{11}, r_{22})) = (297, 22, 66, 62, (10, -1, -1, 0)) \in \Delta^*.$$

By the definition of  $P_{m,r}(t)$ , we obtain

$$P_{m,r}(t) = \{t' \mid t' \equiv ts - (s-1)/8 \pmod{m}, 0 \leq t' \leq m-1, [s]_{24m} \in \mathbb{S}_{24m}\}.$$

We readily verify  $P_{m,r}(t) = \{62\}$ . Setting

$$r' = (r'_1, r'_2, r'_3, r'_6, r'_{11}, r'_{22}, r'_{33}, r'_{66}) = (4, 2, 0, 0, 0, 1, 0, 0).$$

Now let

$$\gamma_\delta = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}.$$

It follows by [7, Lemma 2.6] that  $\{\gamma_\delta : \delta \mid N\}$  contains a complete set of representatives of the double coset  $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ . Since all these constants satisfy the assumption of Lemma 2.2, we obtain the upper bound  $\lfloor v \rfloor = 88$ . Through a similar process, one may see the the upper bound  $\lfloor v \rfloor$  for the  $p_2(297n + 161)$  case is also 88. By Lemma 2.2, we only need to verify terms up to this bound.

Now we will complete our proof with the help of *Mathematica*. We first note that

$$p_2(n) = \sum_{\substack{i+2j=n \\ i,j \geq 0}} p(i)p(j).$$

Note also that  $p(n)$  is computable by the *Mathematica* function `PartitionsP`. One readily verifies that (2.1) holds for both  $t = 62$  and  $161$  when  $n \leq 88$ . This ends the proof of Theorem 2.1.  $\square$

**Remark 2.1.** It is still natural to ask if there are elementary proofs of the two congruences. Considering the difficulty of finding 11-dissection formulas for some  $q$ -series products, we leave this as an open problem.

### 3. AN ELEMENTARY ALTERNATIVE PROOF FOR CHAN'S CONGRUENCE

Although we fail to give an elementary proof for our Theorem 2.1, we do find an elementary alternative proof for Chan's congruence (1.3).

Note that

$$\begin{aligned} \sum_{n \geq 0} p_2(n)q^n &= \frac{1}{(q; q)_\infty (q^2; q^2)_\infty} = \prod_{n \geq 1} \frac{1}{1 - q^n} \prod_{n \geq 1} \frac{1}{1 - q^{2n}} \\ &= \prod_{n \geq 1} \frac{1 - q^n}{1 + q^n} \left( \prod_{n \geq 1} \frac{1}{1 - q^n} \right)^3. \end{aligned}$$

It is well known that

$$\sum_{n \geq 0} s(n^2)q^{n^2} := 1 + 2 \sum_{n \geq 1} (-q)^{n^2} = \prod_{n \geq 1} \frac{1 - q^n}{1 + q^n};$$

see [2, Chapter 16, Entry 22(i)] and [2, Chapter 16, Eq. (22.4)]. We therefore have

$$p_2(n) = \sum_{\substack{m^2+i+j+k=n \\ m,i,j,k \geq 0}} s(m^2)p(i)p(j)p(k). \quad (3.1)$$

Since  $3n + 2 - m^2 \equiv 1$  or  $2 \pmod{3}$ , at least two of  $i, j, k$ , the solution to

$$i + j + k = 3n + 2 - m^2, \quad (3.2)$$

are distinct. If the pairwise distinct triple  $(i, j, k)$  is a solution to (3.2), then any permutation of  $(i, j, k)$  (viz.,  $(j, k, i)$ , etc.) is a solution to (3.2). If  $i = j \neq k$ , then

$(i, k, i)$  and  $(k, i, i)$  are also solutions to (3.2). We therefore obtain

$$p_2(3n+2) = 6 \sum_{\substack{m^2+i+j+k=3n+2 \\ m \geq 0, i > j > k \geq 0}} s(m^2)p(i)p(j)p(k) + 3 \sum_{\substack{m^2+i+j+k=3n+2 \\ m, i, j, k \geq 0, i=j \neq k}} s(m^2)p(i)p(j)p(k). \quad (3.3)$$

This leads to

**Theorem 3.1** (Chan). *For any nonnegative integer  $n$ ,*

$$p_2(3n+2) \equiv 0 \pmod{3}. \quad (3.4)$$

#### 4. RECURSION FOR THE CUBIC PARTITIONS

We know that the popular recursion of  $p(n)$  links partitions to the divisor function. In this section we wish to show that a similar recursion applies to  $p_2(n)$ . Actually, this is a special case of recursions for 2-color partitions where one of the colors appears only in parts that are multiples of  $k$ . Let  $p_k(n)$  denote the number of such partitions. According to [1], its generating function is

$$\sum_{n \geq 0} p_k(n) q^n = \frac{1}{(q; q)_\infty (q^k; q^k)_\infty}, \quad |q| < 1. \quad (4.1)$$

For properties of  $p_k(n)$ , the reader may refer to [1, 5].

Recall that the recursion for  $p(n)$  is as follows

$$p(n) = \frac{1}{n} \sum_{m=1}^n \sigma(m) p(n-m), \quad (4.2)$$

where  $\sigma(n) = \sum_{d|n} d$ . Let  $\sigma^{(k)}(n)$  denote the sum of the labelled divisors of  $n$ , that is, the multiples of  $k$  have two labels. For example,  $\sigma^{(2)}(4) = 1 + 2_1 + 2_2 + 4_1 + 4_2 = 13$ . Our result is

**Theorem 4.1.** *For any nonnegative integer  $n$ ,*

$$p_k(n) = \frac{1}{n} \sum_{m=1}^n \sigma^{(k)}(m) p_k(n-m). \quad (4.3)$$

*Proof.* Take

$$F(q) = \sum_{n \geq 0} p_k(n) q^n,$$

then

$$qF'(q) = \sum_{n \geq 1} n p_k(n) q^n.$$

Let  $G(q) = 1/F(q) = \prod_{n \geq 1} \{(1 - q^n)(1 - q^{kn})\}$ , we have

$$qF'(q) = -q \frac{G'(q)}{G(q)^2} = -q \frac{G'(q)}{G(q)} F(q). \quad (4.4)$$

Note that

$$-q \frac{G'(q)}{G(q)} = -q (\log G(q))'$$

$$= \sum_{n \geq 1} \frac{nq^n}{1 - q^n} + \sum_{n \geq 1} \frac{knq^{kn}}{1 - q^{kn}}.$$

It is also known that

$$\frac{nq^n}{1 - q^n} = \sum_{k \geq 1} nq^{kn},$$

we therefore obtain

$$-q \frac{G'(q)}{G(q)} = \sum_{n \geq 1} \left( \sum_{d|n} d + \sum_{\substack{d|n \\ k|d}} d \right) q^n = \sum_{n \geq 1} \sigma^{(k)}(n) q^n.$$

Combining it with (4.4), one immediately sees that

$$np_k(n) = \sum_{m=1}^n \sigma^{(k)}(m) p_k(n - m).$$

This ends the proof of Theorem 4.1. □

#### REFERENCES

1. Z. Ahmed, N. D. Baruah, and M. G. Dastidar, New congruences modulo 5 for the number of 2-color partitions, *J. Number Theory* **157** (2015), 184–198.
2. B. C. Berndt, *Ramanujan's notebooks. Part III*, Springer-Verlag, New York, 1991. xiv+510 pp.
3. H.-C. Chan, Ramanujan's cubic continued fraction and an analog of his “most beautiful identity”, *Int. J. Number Theory* **6** (2010), no. 3, 673–680.
4. W. Y. C. Chen and B. L. S. Lin, Congruences for the number of cubic partitions derived from modular forms, arXiv:0910.1263, 15pp.
5. S. Chern, New congruences for 2-color partitions, *Preprint*, arxiv:1511.00246, 7pp.
6. S. Radu, An algorithmic approach to Ramanujan's congruences, *Ramanujan J.* **20** (2009), no. 2, 215–251.
7. S. Radu and J. A. Sellers, Congruence properties modulo 5 and 7 for the pod function, *Int. J. Number Theory* **7** (2011), no. 8, 2249–2259.
8. S. Ramanujan, *Collected papers of Srinivasa Ramanujan*, AMS Chelsea Publishing, Providence, RI, 2000. xxxviii+426 pp.
9. X. H. Xiong, The number of cubic partitions modulo powers of 5 (Chinese), *Sci. Sin. Math.* **41** (2011), no. 1, 1–15.

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